

Math Review

M.1 What math?

Microeconomics can be pretty mathematical, even at the intermediate level. For example, some books use calculus extensively in order to solve numerical optimization problems. Because the focus of this book is on low-data situations, we do not use that much mathematics. (A student could otherwise get a false sense of accomplishment by calculating solutions to all kinds of problems, when in real life he or she will never have the data for such calculations.) We do work through numerical examples, but the goal is to illustrate concepts that can be applied to qualitative decision problems.

Hence, the math in this book is basic—no more than what any student would have been exposed to in high school or in a first-year college course at the latest. That said, if you have not used such tools for several years (and perhaps never really liked them back in high school), then a review is important.

M.2 Functions

We are often interested in the relationship between two variables, such as between price P and demand Q for a good or between output Q and cost C of a firm. We think of one of the variables (the dependent variable) as depending on the other (the independent variable). (If no particular meaning is ascribed to the variables, it is common to denote the independent variable by X and the dependent variable by Y .) The relationship between the two variables is called a *function* or *curve*. (“Function” is the standard terminology in math, but in this text we use “function” only when there are several independent variables; otherwise we use the term “curve”.)

If the independent variable can take on only a few values, then we can specify a function with a table that gives the value of the dependent variable for each value of the independent variable. For example, output and revenue may be related as in Table M.1 (taking into account that, to sell more output, you have to charge a lower price).

Table M.1

Output	Revenue
0	0
1	57
2	108
3	153
4	192
5	225
6	252
7	273
8	288
9	297
10	300
11	297
12	288
13	273

We can also specify a function by a formula. For example, perhaps the demand Q as a function of price P is

$$Q = 2800 - 7P. \quad (\text{M.1})$$

According to this formula, when the price is 100, demand is $2800 - (7 \times 100) = 2100$. Perhaps revenue R as function of output Q is

$$R = 60Q - 3Q^2.$$

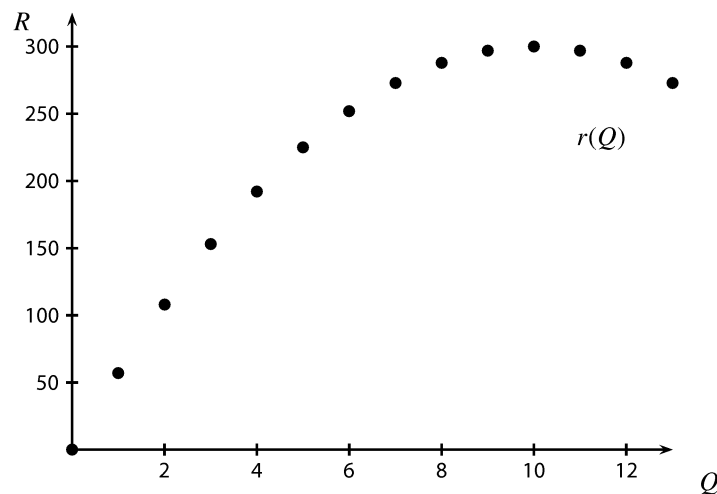
According to this formula, when output is 5, revenue is $(60 \times 5) - (3 \times 5^2) = 300 - 75 = 225$. (This is the formula for the data in Table M.1.)

Often we want to give a function a name. For example, we might denote a demand function by d and a supply function by s . Then $d(P)$ denotes the demand when the price is P and $s(P)$ denotes the supply when the price is P . This allows us to write expressions like: “If the demand curve is d and the supply curve is s , then the equilibrium price P is such that $d(P) = s(P)$.”

M.3 Graphs

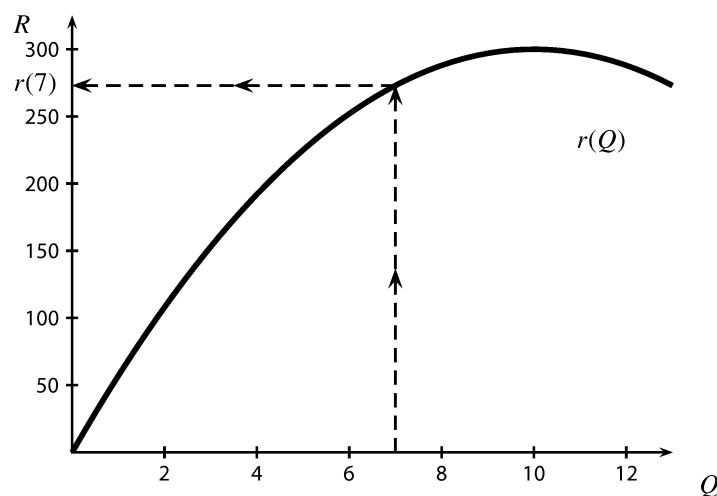
To graph a function, we let the horizontal axis measure values of the independent variable (the “X-axis”) and the vertical axis measure values of the dependent variable (the “Y-axis”). If the data is in table form, then each pair (X, Y) from the table (assuming that X is the independent variable and Y is the dependent variable) is one point on the graph. We plot all the points from the table. Figure M.1 shows the graph of the function in Table M.1.

Figure M.1



If the variables are continuous and we have a functional form, then the graph of the function is a smooth curve. Figure M.2 shows the graph of $r(Q) = 60Q - 3Q^2$.

Figure M.2

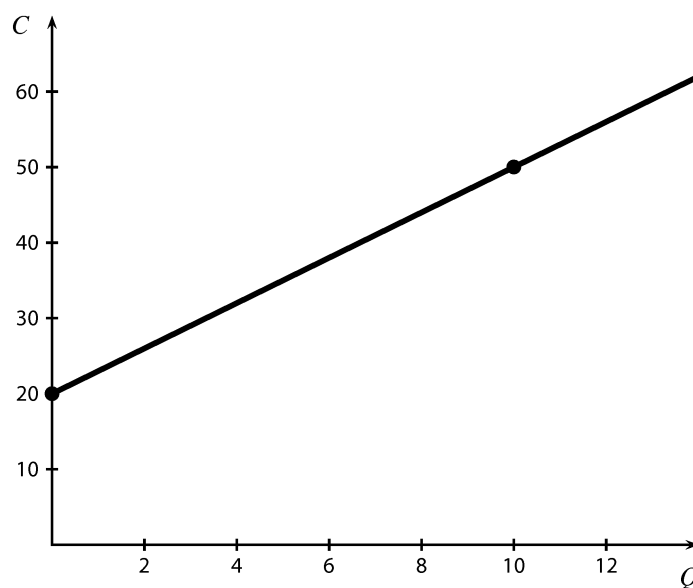


From a graph, you can see the approximate value of the dependent variable for any value of the independent variable. For example, what is $r(7)$? You find 7 on the horizontal

axis, move straight up to the graph of the function, and then look straight left to see what the value is on the vertical axis. This procedure is represented in Figure M.2 by the dashed line; the value of $r(7)$ is approximately 275.

You do not need to be able to draw graphs like the one in Figure M.2, just interpret them. (One can draw a graph like this with Excel or some other software.) You should, however, be comfortable drawing linear functions, such as $Q = 2800 - 7P$ or $C = 20 + 3Q$. The easiest way is to find two points on the curve and then draw a straight line through the two points. Take the function $C = 20 + 3Q$. If $Q = 0$ then $C = 20$; if $Q = 10$ then $C = 50$. Figure M.3 shows these two points, $(0, 20)$ and $(10, 50)$, and the entire graph.

Figure M.3



M.4 Inverse of a function

Let $Q = d(P)$ be a demand function. You can think of a function as a little machine that provides answers. You plug in 7 and get $d(7)$, which is the answer to the question: “what is the demand when the price is 7?”

We can reverse the roles of the variables in a function. Rather than plugging in a price to find a quantity, we can start with a quantity and find the corresponding price. For example, we can ask “for what price is demand equal to 3?” This is called the *inverse* of the function.

Let's see graphically how to read an inverse. Figure M.4 shows the graph of a demand curve $Q = d(P)$. The dashed line represents the procedure of starting with a price of 7 and finding what the demand is at this price. We see that it is 3.

Figure M.4

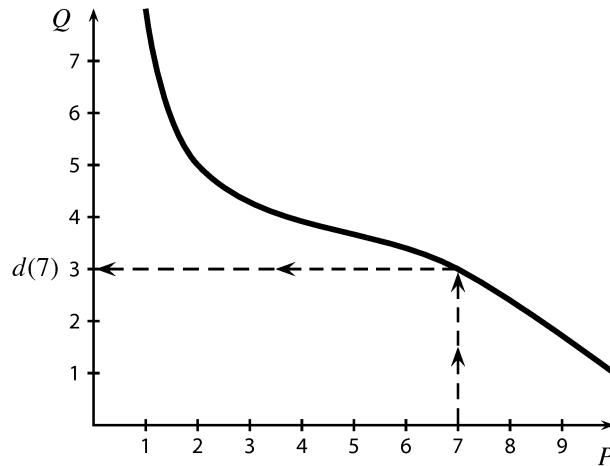
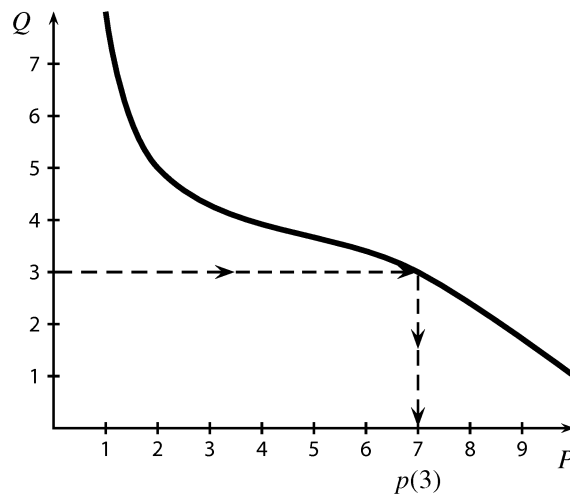


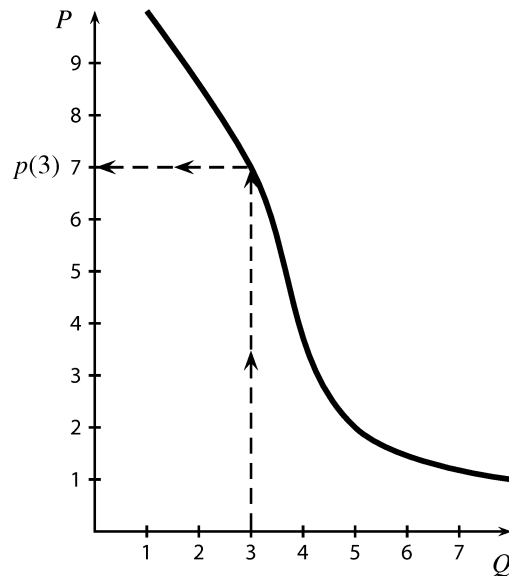
Figure M.5 shows how we can use the same graph to find an inverse. We start with a quantity of 3 on the vertical axis. We go right to the graph of the function and then look down to the value on the horizontal axis. This is the price at which demand is 3.

Figure M.5



The inverse of a function is itself a function. Let $P = p(Q)$ be the inverse of the demand function shown in Figure M.4. Since Q is the independent variable of the function p , mathematical convention is to draw the graph of p with Q on the horizontal axis, as shown in Figure M.6.

Figure M.6



If we need to refer simultaneously to a function and to its inverse, it gets very confusing to flip the graph back and forth. (Visually, the graph in Figure M.4 is obtained by flipping the graph in Figure M.5 along a diagonal line through the origin.) Instead it is easier to keep the graph fixed and to just read it in different directions depending on whether we are working with the function or with its inverse.

M.5 The inverse of a linear function

The only inverses we will actually calculate are for linear functions. Suppose a demand function is $Q = 60 - 5P$, which we can also write as $d(P) = 60 - 5P$. Let $p(Q)$ be its inverse. To find the functional form for $p(Q)$, we solve the equation

$$Q = 60 - 5P$$

for P (that is, we rearrange the equation so that P is alone on the left). Here are the calculations step-by-step:

$$\begin{aligned} Q &= 60 - 5P, \\ 5P &= 60 - Q, \\ P &= 12 - Q/5. \end{aligned}$$

(We added $5P$ to both sides, subtracted Q from both sides, and then divided both sides by 5.) Thus, the inverse can be written $P = 12 - Q/5$ or $p(Q) = 12 - Q/5$.

From now on in this text, we graph demand and supply curves as inverses, with P on the vertical axis and Q on the horizontal axis. Economists started doing this over 120 years ago, not because they were poor mathematicians, but because the inverses of demand and supply curves are used even more often than the demand and supply curves themselves.

Exercise M.1. We first plot a demand curve following *mathematical* convention (with price on the horizontal axis). Then we calculate and plot its inverse.

- a. Draw a graph of the linear demand curve $d(P) = 20 - \frac{1}{3}P$, with price P on the *horizontal* axis and demand on the *vertical* axis. Be sure to label the units on the axes or at least the values of the intercepts.
- b. Now calculate the inverse $P = p(Q)$ of this demand curve. You are solving the equation $Q = 20 - \frac{1}{3}P$ for P . What is $p(9)$?
- c. Graph the inverse demand curve, with quantity on the horizontal axis and price on the vertical axis. Again, label the values of the intercepts.
- d. The general form of a linear demand curve is $d(P) = A - BP$, where A and B are positive numbers. The price at which demand is 0 is $\bar{P} = A/B$, which we call the *choke price*. Write the general form for the inverse demand curve, using the coefficients \bar{P} and B .

Hint: You solve $Q = A - BP$ for P . If you arrange the formula the right way, the term A/B appears; replace it by \bar{P} .

M.6 Some nonlinear functions

We work (infrequently) with two kinds of nonlinear functions: exponents and logarithms.

Examples of exponential functions are $Q = K^{1/2}L^{1/4}$ and $Q = 3P^{-2}$. Note that $P^{-2} = 1/P^2$ and so the second function could also be written $Q = 3/P^2$.

Logarithms are used in the following way. When we take log of a mathematical expression, multiplication becomes addition and exponents become multiplication. Here are some examples:

$$\log(8X) = \log 8 + \log X,$$

$$\log(X^7) = 7 \log X,$$

$$\log(4X^2Y^{-3}) = \log 4 + 2 \log X - 3 \log Y.$$

Thus, if $Q = 3P^{-2}$, then, by taking the log of both sides, we get $\log Q = \log 3 - 2 \log P$. This function is now linear if we think of the variables as $\log Q$ and $\log P$.

Exercise M.2. Expand the following, as above.

$$\log(12P) =$$

$$\log(P^{-1/2}) =$$

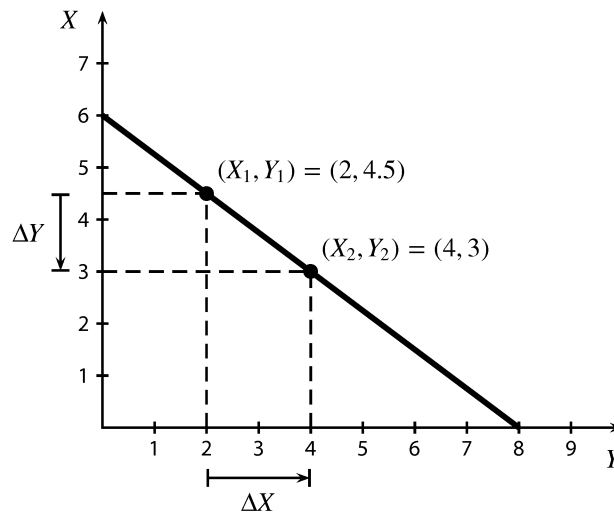
$$\log(8P^{-3}I^{0.75}) =$$

M.7 Slope of a linear function

The slope of a linear function measures how much the dependent variable changes per unit change in the independent variable. Slope is important to us because we study how profit, for example, changes if we adjust actions by a small amount.

Slope may be denoted $\Delta Y / \Delta X$, where ΔY means “change in Y ” and ΔX means “change in X ” between two points on the line. (This ratio is the same for any two points.) In Figure M.7, comparing the two points $(2, 4.5)$ and $(4, 3)$ on the graph, we have $\Delta X = 2$ and $\Delta Y = -1.5$. Hence, the slope is $-1.5/2 = -3/4$.

Figure M.7



A line that slopes down has negative slope; a line that slopes up has positive slope. The steeper is the line, the greater is the magnitude of the slope (the opposite is true when we are working with an inverse and graph the dependent variable on the horizontal axis).

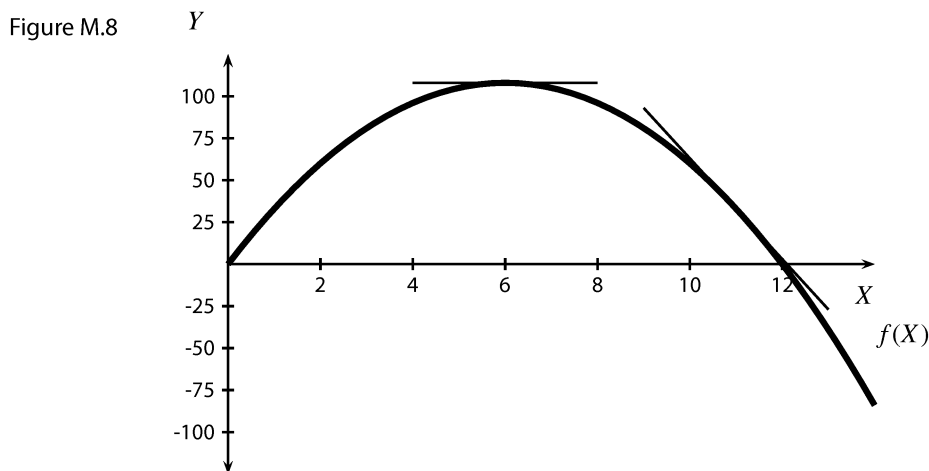
If we have the formula for a line, the slope is the coefficient on the independent variable. For example, the slope of $C = 20 + 3Q$ is 3. If Q goes up by 2, then cost goes up by 6. If Q goes down by -0.4 then cost goes down by $3 \times (-0.4) = -1.2$.

Exercise M.3. Consider the function $C = 30 + 4Q$. Identify the slope in two different ways: (a) from the coefficient on the independent variable Q ; and (b) by calculating $\Delta C / \Delta Q$ for two points on the graph. Verify that you get the same answer.

M.8 Slope of a nonlinear function

Graphically

The slope of a nonlinear smooth function at a particular point $Y = f(X)$ is equal to the slope of the line tangent to the graph of the function. It is denoted by dY/dX or $f'(X)$ or $mf(X)$. (The notation $f'(X)$ is common in math but we use $mf(X)$ in the main text.)



Consider Figure M.8, which shows the graph of a function $Y = f(X)$. The lines tangent to the curve illustrate the slope at those points. Slope is positive at first but the curve gets less steep as X increases. Thus, slope is falling, until it is zero at $X = 6$. The curve then slopes downward, meaning that slope is negative. Since it gets steeper, slope is getting more negative as X increases; that is, slope continues to fall.

For a nonlinear function like the one in Figure M.8, the slope gives an approximate measure of the rate at which the value of the function changes for small changes in X . For example, the slope is -30 at $X = 11$. This means that, if X increases by 0.1 from $X = 10$ to $X = 10.1$, then the value of the function changes by approximately $-30 \times 0.1 = -3$.

Slope as the derivative of a function

To calculate the slope of a nonlinear function, we find its derivative. This is a technique from calculus. The derivatives we use are pretty simple:

- The derivative of a constant function like $f(X) = 5$ is 0; the graph of the function is a flat line.
- The derivative an exponential function of the form $f(X) = AX^B$ is $f'(X) = BAX^{B-1}$. That is, we multiply the function by the exponent and reduce the exponent by 1. Here are some examples:

$f(X)$	$f'(X)$
X^4	$4X^3$
$3X^5$	$15X^4$
$10X^{-2}$	$-20X^{-3}$
$4X$	4

The last example is just the formula for the slope of a linear function; we use the fact that a number to the 0 power, such as X^0 , is equal to 1.

- The derivative of the sum of several terms is the sum of the derivatives of the terms. For example, if

$$f(X) = 4 + 10X - 3X^2,$$

then

$$f'(X) = 10 - 6X.$$

Exercise M.4. Calculate the derivatives of the following functions.

- a. $R = 60Q - 3Q^2$.
 - b. $d(P) = 3P^{-2}$.
 - c. $f(X) = 18 - 5X + X^2$.
-