MBA Business Foundations, Quantitative Methods: Session Two

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Today

Exponents

- Essential for analysis of interest rates and growth.
- Denotes repeated multiplication of the same quantity

Exponent

$$x^{t} = \underbrace{x \cdot x \cdot \ldots \cdot x}_{\text{t times}}$$

Base

Exponents

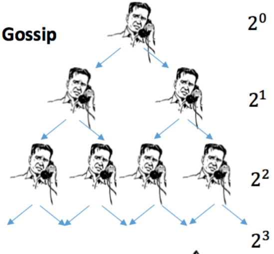
Examples:

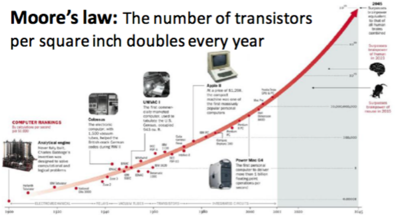
- 3^4
- 1.5^2
- $\left(\frac{1}{2}\right)^2$
- $(a+2)^2$

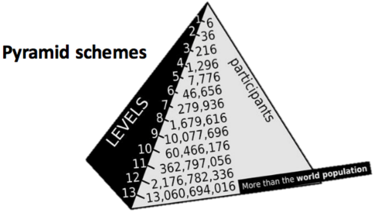
Solutions:

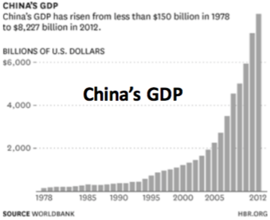
- $3^4 = 81$
- $1.5^2 = 2.25$
- $\left(\frac{1}{2}\right)^2 = \left(\frac{1}{4}\right)$
- $(a+2)^2 = a^2 + 4a + 4$ (what kind of function?)

Some examples of exponential phenomena









Rules for exponents

Products	$b^c \cdot b^d = b^{c+d}$
Powers	$(b^c)^d = (b^d)^c = b^{c \cdot d}$
Negative exponents	$b^{-c} = \frac{1}{b^c}$
Quotients	$\frac{b^c}{b^d} = b^{c-d}$
Zero power	$b^0 = 1$
Roots	$\sqrt[n]{b} = b^{1/n}$

The formulas in practice

Simplify the following:

- $(3^4)^2 =$
- $\frac{6^2}{6^5} =$
- $3^0 =$
- $27^{2/3} =$
- $\left(\frac{x}{y}\right)^3 \cdot \left(\frac{x}{z}\right)^{-2}$
- $\frac{x^3 y^2}{x^5 y^{-2}}$
- $\frac{24x^5 y^3 z^7}{6x^3 y^2 z^4}$

Solutions

- $(3^4)^2 = 3^8$
- $\frac{6^2}{6^5} = 6^{2-5} = 6^{-3} = \frac{1}{6^3}$
- $3^0 = 1$
- $27^{2/3} = \sqrt[3]{27^2}$
- $\left(\frac{x}{y}\right)^3 \cdot \left(\frac{x}{z}\right)^{-2} = \left(\frac{x}{y}\right)^3 \cdot \left(\frac{z}{x}\right)^2 = \frac{x^3 z^2}{y^3 x^2} = \frac{x z^2}{y^3}$
- $\frac{x^3 y^2}{x^5 y^{-2}} = \frac{x^3 y^2 y^2}{x^5} = x^{-2} y^4 = \frac{y^4}{x^2}$
- $\frac{24x^5 y^3 z^7}{6x^3 y^2 z^4} = 4x^2 y z^3$

Simple interest

- I: interest income in one period
- P: capital to invest
- r: interest rate per period
- n: number of periods invested
- $P = \$1,000$, $r = 4\%$, what is I after one year?
- Solution: $\$40$
- if $n = 3$, what is I?
- Solution $\$120$
- In general, $I = P \cdot r \cdot n$

Compound interest, part 1

- (Capital increases by interest every period)
- You invest $\$1$ at an annual interest rate $r = 4\%$.
- After year 1: $1 + 1(0.04) = 1.04$
- After year 2: $1.04 + 1.04(0.04) = 1.04(1 + 0.04) = 1.04^2$
- After year 3: $1.04^2 + 1.04^2(0.04) = 1.04^2(1 + 0.04) = 1.04^3$
- After year t: 1.04^t

  Some notation:
- $P =$ present amount
- $A =$ final amount
- $r =$ interest rate
- $t =$ number of years money is invested.
- General formula: if compounding annually, $A = P(1 + r)^t$

Compound interest, part 2

Compounding could be done:

- Yearly $\rightarrow$ rate/period $= r$
- Semi-annually $\rightarrow$ rate/period $= r/2$
- Quarterly $\rightarrow$ rate/period $= r/4$
- Monthly $\rightarrow$ rate/period $= r/12$
- Some more notation: n number of periods per year
- A more general formula: $A = P\left(1 + \frac{r}{n}\right)^{tn}$

Compound interest, part 3

A general inverse formula: if we know the final amount A, the interest rate r, the time money is invested t and compounding periods per year, n, we can calculate the principal P.

$$P = A\left(1 + \frac{r}{n}\right)^{-tn}$$

Examples on interest rates, part 1

- $P = \$1000$, $r = 4\%$, $t = 3$ years
- Compare the final amount A for,
- Simple interest
- Compounded annually
- Compounded semi-annually
- Compounded quarterly
- Compounded monthly

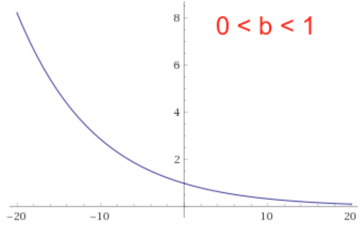
  Answers:
- $1000 \cdot 0.04 \cdot 3 = 120 \rightarrow 1000 + 120 = 1120$
- $A = P(1+r)^t \rightarrow 1000(1+0.04)^3 = 1124.87$
- $A = P(1+r/n)^{tn} = 1000(1+0.04/2)^{3\cdot 2} = 1126.12$
- $A = P(1+r/n)^{tn} = 1000(1+0.04/4)^{3\cdot 4} = 1126.83$
- $A = P(1+r/n)^{tn} = 1000(1+0.04/12)^{3\cdot 12} = 1127.27$

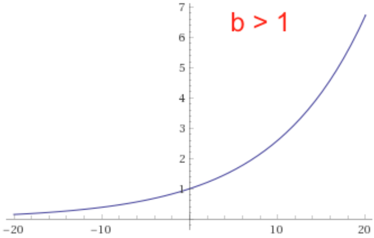
Examples on interest rates, part 2

- Problem 1: I borrowed $\$2,000$ for 5 years at $r = 8\%$, compounded quarterly. How much do I have to pay back at the end of the term?
- Problem 2: I put my money in a savings account at $r = 6\%$ which is compounded semi-annually and received $\$530.45$ at the end of the year. How much did I put in at the beginning?
- Solution 1: $2000(1 + 0.08/4)^{5 \cdot 4} = 2971.89$
- Solution 2: $P = A(1 + r/n)^{-tn} = 530.45(1 + 0.06/2)^{-2} = 500$

Exponential functions

$f(x) = b^x$ where $b > 0$, b is the base, x is the exponent.
(Check: is x^2 exponential? Why or why not?)



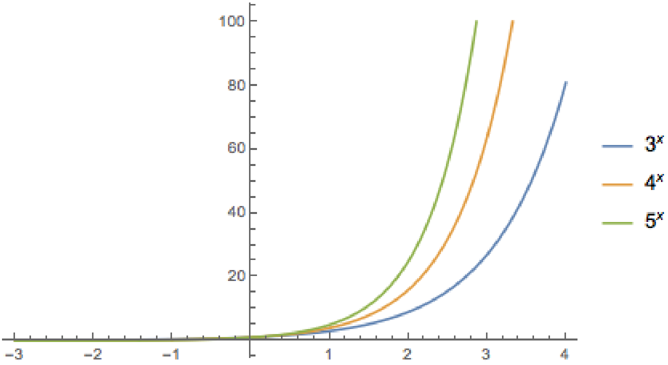


Exponential functions

- All the curves pass through the point $(x, y) = (0, 1)$.
- The exponential functions are always above the $f(x) = 0$ horizontal line. In fact, that line is an asymptote.
- $f(x) \to 0$ as $x \to -\infty$ (when $b > 1$), and $f(x) \to 0$ as $x \to \infty$ when $0 < b < 1$.
- Can we have $b < 0$? Consider $f(x) = -4^x$. What is $f(2)$?, $f(-3)$?, $f(1/2)$?
  (hint: $\sqrt{-4} = 2i$).

Exponential functions

- If $b > 1$, the curve becomes steeper as b increases
- if $0 < b < 1$, it is the other way around, the curve becomes steeper as the base gets closer to 0.



Practice

Match each equation with the graph of f, g, h, k:

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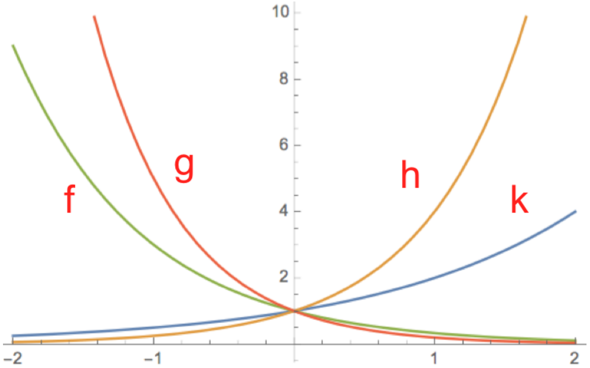
Match each equation with the graph of f, g, h, k:

(A) $f(x) = 2^x$

(B) $f(x) = (0.2)^x$

(C) $f(x) = 4^x$

(D) $f(x) = (1/3)^x$



A generalized example

- Exponential functions can be generalized to $f(x) = ab^x$ where a is now a scaling constant of the function.
- Ex: Compound interest when compounded annually.

  Recall it is given by $A = P(1+r)^t$. Here $a = P$, $b = (1+r)$, and $x = t$.
- Ex: Compound interest when compounded n times per year.

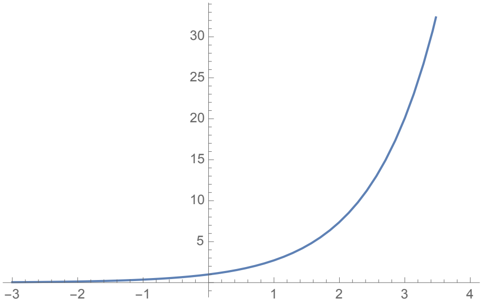
  Given by $A = P\left(1 + \frac{r}{n}\right)^{tn}$
- Why is this an exponential function of the form $f(x) = ab^x$?

  $A = P\left[\left(1 + \frac{r}{n}\right)^n\right]^t$

  Here $a = P, b = \left(1 + \frac{r}{n}\right)^n, x = t$.

A very special case

- $f(x) = e^x$, where $e \approx 2.71828$, named after Leonhard Euler.



- Examples:
- Finance: continuous compounding: $A = P \cdot e^{rt}$
- Economics: growth rate: $e^{0.03t}$
- Probability: exponential families (includes normal distribution!)

Building intuition

A hypothetical:

Building intuition

A hypothetical: the infinitely evil loan shark...

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Building intuition

Suppose you borrow one dollar from an ordinary loan shark that charges 100% interest, n times per year. Then after the year, you will owe:

$$\left(1+\frac{1}{n}\right)^n$$

- In particular, after 1 year:

  For $n = 1$, you will owe 2.

  For $n = 2$, you will owe 2.25

  For $n = 3$, you will owe 2.37.

  For $n = 100$, you will owe 2.704.
- The infinitely evil loan shark: $n = \infty$,

Building intuition

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  For $n = 100$, you will owe 2.704.

- The infinitely evil loan shark: $n = \infty$, owe e.
- That is, $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e \approx 2.7182818\ldots$!!
- Your debt to the infinitely evil loan shark is finitely bounded by e!

Building intuition

- If you borrow A_0, after t years, you will owe,

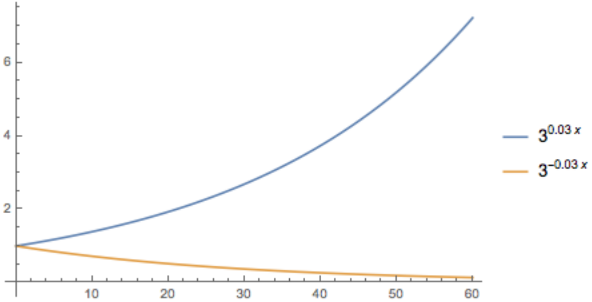
$$A = \lim_{n \to \infty} A_0 \left[\left(1 + \frac{1}{n} \right)^n \right]^t = A_0 e^t$$

- Now suppose the loan shark charges $(k \cdot 100)\%$ interest continuously.
- Reminder: A_0 is your starting point, $(k \cdot 100)\%$ is the growth rate, t is the number of periods, and $e = (1 + 1/n)^n$ as n gets arbitrarily large.
- Then in general you will owe

$$A = A_0 e^{kt}$$

- The growth model involving e appears naturally as the continuous generalization of the compounding interest formula.

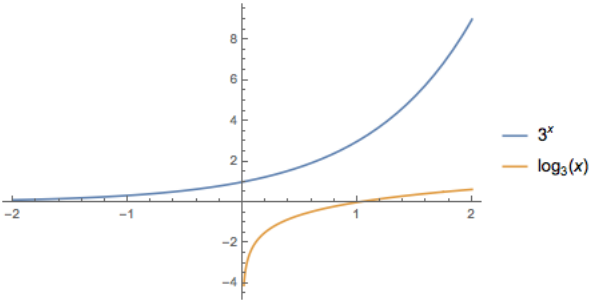
Example: growth/decay



- $A = A_0 e^{kt}$, where $A =$ ending value, $A_0 =$ initial value, t is elapsed time, and k is the growth/decay rate.
- $k > 0$, the amount is increasing (growing); $k < 0$, the amount is decreasing (decaying).
- Ex: bacteria grow continuously – i.e., they do not "wait" and then all at once reproduce in the next period.

Logarithmic functions

- A logarithmic function is the inverse of an exponential function
- If $b^x = c$ then $\log_b(c) = x$
- Natural log: if $e^x = c$ then $\ln(c) = x$ where $\ln x = \log_e x$

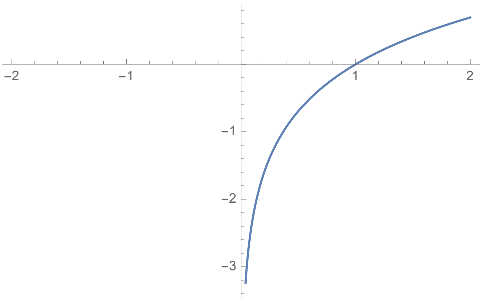


Logarithmic functions

- There are no logs of zero or negative numbers $(x > 0)$ (Why?).
- If $\log_b(-k) = c$ then $b^c = -k$.
- Logs of numbers less than one are negative.
- All curves pass through the point $(x, y) = (1, 0)$.
- When x tends to 0 in positive value, $f(x)$ is higher and higher in negative value.
- The vertical line at $x = 0$ is an asymptote: a straight line which the graph approaches but never touches.

Operations on logs

- $\log_b(b^x) = x$
- $b^{\log_b(x)} = x$
- $\log_b(c \cdot d) = \log_b c + \log_b d$
- $\log_b \frac{c}{d} = \log_b c - \log_b d$
- $\log_b(c^d) = d \cdot \log_b c$
- $\log_b(b) = 1$
- $\log_b 1 = 0$



Example: Bacteria growth

- Problem: A strain of bacteria growing on your desktop doubles every 5 minutes. Assuming that you start with only one bacterium, how many bacteria could be present after 1.5 hours? Hint: $\log(e^x) = x$.

$$
\begin{aligned}
& A = A_0 e^{kt} \\
& \quad \rightarrow 2 = 1 \cdot e^{k \cdot 5} \\
& \quad \rightarrow 2 = e^{5k} \\
& \quad \rightarrow \log 2 = \log e^{5k} \\
& \quad \rightarrow \log 2 = 5k \\
& \quad \rightarrow \frac{\log 2}{5} = k \\
& \quad \rightarrow k = 0.139 \\
& \quad \rightarrow A = A_0 e^{0.139 \cdot 90} \\
& \quad \rightarrow A = 1 \cdot e^{0.139 \cdot 90} = 271,034!
\end{aligned}
$$

Example: college savings

The parents of a newborn child want to have $\$25,000$ for the child's college education when she is 18.

At what rate of interest, compounded continuously, must $\$10,000$ be invested now to achieve this goal?

Solution: college savings

$$A = A_0 e^{rt}$$
$$25000 = 10000e^{r18}$$
$$\ln(25000) = \ln(10000 \cdot e^{r18})$$
$$\ln(25000) = \ln(10000) + \ln e^{r18}$$
$$\ln(25000) = \ln(10000) + r18$$
$$\ln(25000) - \ln(10000) = r18$$
$$\ln(\frac{25000}{10000}) = r18$$
$$r = 0.05 = 5\%$$

Practice examples

Practice:

- Ex 1: $\log_5(4x+11)=2$
- Ex 2: $\log_2(x+5)-\log_2(2x-1)=5$
- Ex 3: $\log_8(x)+\log_8(x+6)=\log_8(5x+12)$

  Hint: get into quadratic form, find positive root
- Ex 4: $\log_6(x)+\log_6(x-9)=2$
- Ex 5: $\ln(10)-\ln(7-x)=\ln(x)$

Ex 1 Solution

$$\begin{aligned} \log_5(4x+11) &= 2 \\ 4x+11 &= 5^2 \\ x &= 7/2 \end{aligned}$$

Ex 2 Solution

$$\log_2(x+5) - \log_2(2x-1) = 5$$

$$\log_2\left(\frac{x+5}{2x-1}\right) = 5$$

$$\left(\frac{x+5}{2x-1}\right) = 2^5 = 32$$

$$x+5 = 32(2x-1)$$

$$x = 37/63$$

Ex 3 Solution

$$
\begin{aligned}
\log_8(x) + \log_8(x+6) &= \log_8(5x+12) \\
\log_8(x(x+6)) &= \log_8(5x+12) \\
x(x+6) &= 5x+12 \\
x^2 + 6x &= 5x+12 \\
x^2 + x - 12 &= 0 \\
(x-3)(x+4) &= 0 \rightarrow x = 3
\end{aligned}
$$

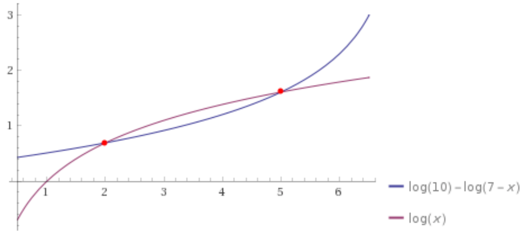
Ex 4 Solution

$$\begin{aligned} \log_6(x) + \log_6(x-9) &= 2 \\ x(x-9) &= 36 \\ (x+3)(x-12) &= 0 \\ x &= 12 \end{aligned}$$

Ex 5 Solution

$$\ln(10) - \ln(7-x) = \ln(x)$$
$$\ln\left(\frac{10}{7-x}\right) = \ln x$$
$$\left(\frac{10}{7-x}\right) = x$$
$$10 = x(7-x)$$
$$10 = 7x - x^2$$
$$x^2 - 7x + 10 = 0$$
$$(x-5)(x-2) = 0$$
$$\rightarrow x = 2, x = 5$$

Plot:



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