MBA Business Foundations, Quantitative Methods: Session Four

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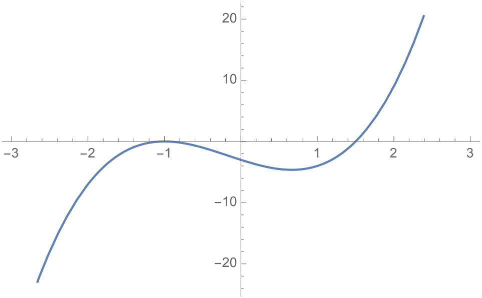
Today

Analyzing functions with derivatives

- We are given some function $f(x)$ and want to know something about its behavior at x_1.
- Find $f'(x)$.
- Find $f'(x_1)$.
- If $f'(x_1) > 0$ the function is increasing "at that point".
- $f'(x_1) < 0$ the function is decreasing "at that point".
- if for all $x, f'(x) > 0$ the function is increasing.
- if for all $x, f'(x) < 0$ the function is decreasing.
- $f'(x_1) = 0$ the function is at a maximum or minimum (most likely).

Example

Consider the function $f(x) = 2x^3 + x^2 - 4x - 3$



- Find $f'(x)$.
- Evaluate x at $(-2, -1, 0, 1, 1.5)$.

Solution

- $f'(x) = 6x^2 + 2x - 4$
- $f'(-2) = 16$
- $f'(-1) = 0$
- $f'(0) = -4$
- $f'(1) = 4$
- $f'(1.5) = 12.5$

Second derivatives

- Consider a function which takes time as its input and gives us a car's distance traveled as its output.
- The first derivative of such a function corresponds to the car's velocity.
- The second derivative would be the derivative of the derivative.
- This corresponds to the car's acceleration.
- It measures how the rate of change is itself changing.
- Graphically, this corresponds to a function's curvature – degree of concavity/convexity.
- Formally, there is nothing new – to take the second derivative, treat the derivative function as your original function, and apply the rules from last class!

Concavity/convexity

- A function is called convex on an interval $[a, b]$ if the line segment between any two points on the graph of the function over that interval lies above or on the graph. Ex: x^2.
- If such line segment is below the graph of the function, it is concave.
- Important wherever "marginal" values are relevant – utility, revenue, economies of scale, etc.
- We denote the second derivative of $f(x)$ as $f''(x)$ or

$$\frac{d}{dx}\frac{d}{dx}f(x) = \frac{d^2}{dx^2}f(x)$$

- If $f''(x) > 0$ for all x in the interval $[a, b]$, then $f(x)$ is convex on $[a, b]$. Ex: x^2.
- If $f''(x) < 0$ for all x in the interval $[a, b]$, then $f(x)$ is concave on $[a, b]$. Ex: $x^{1/2}$.

Maximum/minimum of a function

To find potential max-min points of a function $f(x)$:

- Compute the (first) derivative $f'(x)$
- Solve the equation $f'(x) = 0$. The points x^* obtained are possible candidates for maxima/minima.

  $\longrightarrow$ First Order Condition (FOC)
- Compute the second derivative $f''(x)$
- If $f''(x^*) > 0$ then x^* is a (local) minimum

  If this is true for all x, then global minimum
- If $f''(x^*) < 0$ then x^* is a (local) maximum

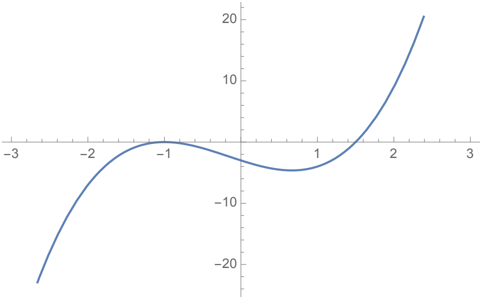
  If this is true for all x then global maximum

  $\longrightarrow$ Second Order Condition (SOC)

Example

Identify the local maximum/minimum of the function:

$$f(x) = 2x^3 + x^2 - 4x - 3$$



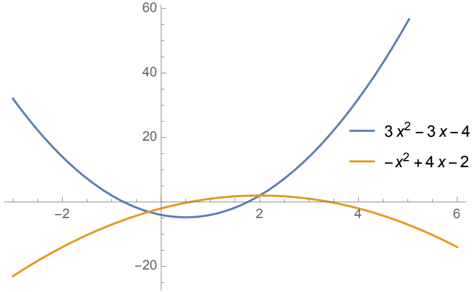
Solution

- $f(x) = 2x^3 + x^2 - 4x - 3$
- $f'(x) = 6x^2 + 2x - 4$
- $f''(x) = 12x + 2$
- FOC: $6x^2 + 2x - 4 = 0 \rightarrow x^* = -1, \text{ and } x^* = 2/3$.
- SOC:

  $12(-1) + 2 < 0$ (-1 is a maximum)

  $12(2/3) + 2 > 0$, (2/3 is a minimum).

Application to quadratic function



- $f(x) = ax^2 + bx + c$
- $f'(x) = 2ax + b \rightarrow$ FOC: $2ax + b = 0 \rightarrow x_0 = \frac{-b}{2a}$
- $f''(x) = 2a$
- if $a > 0$, $x_0 = \frac{-b}{2a}$ is global minimum (b/c $f''(x) = 2a > 0$)
- if $a < 0$, $x_0 = \frac{-b}{2a}$ is global maximum (b/c $f''(x) = 2a < 0$)

Application of derivatives in economics

- Consider profit (p) as a function of advertising cost (c).

  $$p = f(c) = -c^2 + 3c - 2$$

  At what level of advertising will the profit be maximized?

- Consider a demand function, with price (p) and quantity demanded (q).

  $$p = f(q) = 120 - 4q$$

  Write revenue as a function of quantity demanded.

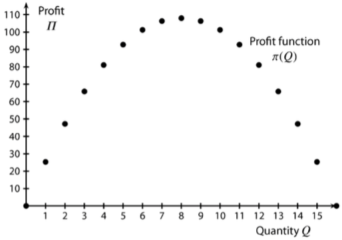
  To maximize revenue, how many units should we sell?

  Which price should we set?

Solutions

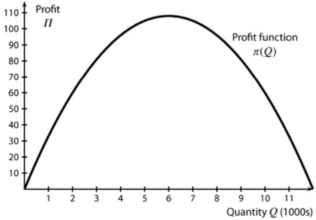
- Problem 1: $\arg\max f(c) = 3/2$ (you may often see this notation).
- Problem 2:
  a. Revenue $=$ price $\times$ quantity $= -4q^2 + 120q$
  b. $-4q^2 + 120q = 0 \rightarrow q^* = 15$ units
  c. $f(q^*) = 120 - 4 \times 15 = 60 \rightarrow p^* = 60$

Marginal profit



- Discrete case, where $\pi(q)$ is profit given q units and $m\pi(q)$ marginal profit at q : $m\pi(q) = \pi(q+1) - \pi(q)$
- "Profit earned by producing one unit above q"
- Note this is *also* the rate of change of $\pi(q)$ at $q \rightarrow \frac{\pi(q+1)-\pi(q)}{q+1-q}$
- Marginal profit is given by the slope of the profit function at q.

Marginal profit



- Continuous case: $m\pi(q) = \pi'(q)$

  "Profit earned by producing *a little more* than q"

  [What is a little more?]

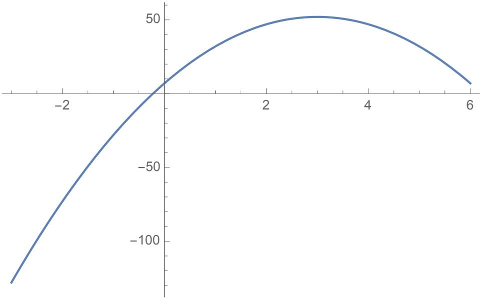
Bonus example: marginal revenue

- Given the following demand function,

$$p = -5q^2 + 30q + 7$$

- find the marginal revenue function, where price $= p$ and quantity $= q$.
- What is the marginal revenue at $q = 2, 3, 5$?
- For which q do we maximize revenue?

Solution



- $f'(q) = -10q + 30$
- $f'(2) = 10, f'(3) = 0, f'(5) = -20$
- $\arg\max f(q) = 3$

Case discussion

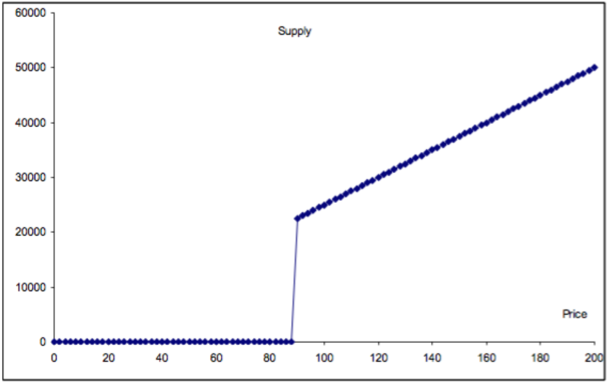
Motorycle Helmets with Bluetooth (B) : Production

Revisit 1d

- Find produced quantity as a function of market price – how much will the firm supply at different price levels.
- $\mathrm{pr}(q) = \mathrm{revenue} - \mathrm{cost} = p \times q - (1,000,000 + 0.002q^2)$
- Profit is maximized when $\mathrm{pr}'(q) = 0 \to p = 0.004q \to q = p/0.004 \to q = 250p$
- So the actual value of profit at different price levels is $p(250p) - (1,000,000 + 0.002(250p)^2) = 125p^2 - 1,000,000$
- So profit is positive if $125p^2 - 1,000,000 > 0 \to p^2 > 8,0000 \to p > 89.44$
- So the supply function (!!) is:

$$s(p) = \begin{cases} 250p, & \text{if } p > 89.44 \\ 0, & \text{if } p < 89.44 \end{cases}$$

Revisit 1d



- If $p < 89.44$ the firm is better off by shutting down and producing zero. If $p > 89.44$ the firm is better off by producing $250p$. At 89.44, the firm is indifferent.

Statistics

Walmart handles more than 1 million customer transactions every hour.

YouTube users upload 48 hours of new video every minute of the day.

Brands and organizations on Facebook receive 34,722 likes every minute of the day.

- How to visualize this data?
- How to understand the key components of this data?

→ goal of descriptive statistics

The R statistical programming language

- For the statistics section, we will periodically use the statistical programming language R to perform some basic operations.
- R is an open source language, written in C and Fortran.
- It was initially developed at Bell Labs, where it was known (creatively) as S.
- Today it is the most widely used language among statisticians.
- You can download a free distribution on the R project web page (www.r-project.org/) together with RStudio (rstudio.com) which is the leading IDE for R (basically a gui).
- In this class, however, we will use the online implementation rdrr: https://rdrr.io/snippets/
- You do not need to download anything!
- Does anyone know why it is called rdrr? See here.

Measures of location

"If I was allowed one number to describe my dataset, what would it be?"

- Three notions:
- Mean: average value, $\mathrm{E}[X] = \frac{1}{n}[x_1 + x_2 + ... + x_n]$

  R: mean()
- Median: the middle value when the dataset is ordered from smallest to largest

  R: median()
- Mode: the value with highest frequency

  R: mode.insead()

  But we have to write the mode function first:

```
mode.insead <- function(x) {
  ux <- unique(x)
  ux[which.max(tabulate(match(x, ux)))]
}
```

- Feel free to copy/paste

Measures of location

- Go to: https://rdrr.io/snippets/
- Consider:

  ```
  mydata <- c(1, 2, 3, 3, 4, 5, 6)
  ```
  [the simplest type of data structure in R]
- Find the mean, median, and mode of mydata. Hint: for mode, you will have to write in the function first.
- Here it is again:

  ```
  mode.insead <- function(x) {
    ux <- unique(x)
    ux[which.max(tabulate(match(x, ux)))]
  }
  ```
- Solution:

  ```
  mode.insead <- function(x) {
          ux <- unique(x)
          ux[which.max(tabulate(match(x, ux)))]
  }

  mydata <- c(1, 2, 3, 3, 4, 5, 6)

  print(c(mean(mydata), median(mydata), mode.insead(mydata)))
  [1] 3.428571 3.000000 3.000000
  ```
- Note: We instruct R to print a vector just to be concise. You *can* just plug in mean, mode, and median and run it.

Example

Given the following data:

$$10, 3, 2, 15, 1, 3, 4, 5, 8, 2, 12, 20, 3, 5, 10$$

compute the mean, median and mode by hand, then verify in R.

Example

Given the following data:

$$10, 3, 2, 15, 1, 3, 4, 5, 8, 2, 12, 20, 3, 5, 10$$

compute the mean, median and mode by hand, then verify in R.

Solution:

```
mode.insead <- function(x) {
       ux <- unique(x)
       ux[which.max(tabulate(match(x, ux)))]
}

mydata <- c(10,3,2,15,1,3,4,5,8,2,12,20,3,5,10)

print(c(round(mean(mydata),2), round(median(mydata),2),
round(mode.insead(mydata),2)))
[1] 6.87 5.00 3.00
```

We have also instructed R to round the output to two decimal points.

Measures of dispersion

How spread out is my data?

- Maximum, minimum, range

  R: max(), min(), max() - min()
- Variance

  R: var.insead()

  ```
  var.insead = function(x){var(x)*(length(x)-1)/length(x)}
  ```
- Standard deviation

  R: sqrt(var.insead())

Variance and standard deviation

- We will often use μ to denote the mean, and σ to denote standard deviation.
- Variance (when all outcomes are equally likely):

$$\mathrm{Var} = \frac{1}{n}\sum_{i=1}^{n}(x-\mu)^2$$

- Standard deviation: $\sigma = \sqrt{\mathrm{Var}}$

  [So: $\mathrm{Var} = \sigma^2$]
- Rougly speaking, 95% of the data will be contained in the interval spanning ± 2 standard deviations from the mean.

Exercise

Consider the following data, drawn from a uniform distribution in R using the command round(runif(10)*10, 0):

$$8, 1, 2, 4, 8, 6, 2, 6, 9, 2$$

Find mean, variance, and standard deviation. Note:

```
var.insead = function(x)
    {var(x)*(length(x)-1)/length(x)}
```

Solution

```
var.insead = function(x)
    {var(x)*(length(x)-1)/length(x)}

mydata <- c(8,1,2,4,8,6,2,6,9,2)

mean(mydata)

var.insead(mydata)

sqrt(var.insead(mydata))

[1] 4.8
[1] 7.96
[1] 2.821347
```

Today

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